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An alternative to the Hamilton–Jacobi approach in classical mechanics

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Abstract. A time-dependent function on the phase space, the *phase action*, is introduced, which is related to the Hamilton principal function by means of the Legendre transformation. The phase action is the (unique) solution of the Cauchy problem for a first-order partial differential equation, analogous to the Hamilton–Jacobi equation. The result is a manifestly invariant phase-space formalism. General properties of the phase action are analysed; in particular, the symmetry under time inversion and the continuous group property. Some examples are considered: the anisotropic oscillator, free motion, the multi-dimensional rotator, and the Kepler problem in Fock variables. As application of the formalism, an invariant perturbation theory is developed. The relation to semiclassical methods in the quantum theory is briefly discussed. A generalised dynamics on a manifold with non-flat symplectic structure is considered in the appendix.

1. Introduction

The principles of classical dynamics are thoroughly investigated, and they are presented in a number of brilliant books (e.g. Goldstein 1957, Synge 1960, Lanczos 1962, Pars 1964, Arnold 1974). It is known, in particular, that to formulate the dynamics one may start from the Hamilton–Jacobi equation

$$\partial A/\partial t + H(\boldsymbol{Q}, \partial A/\partial \boldsymbol{Q}; t) = 0. \tag{1.1}$$

From any complete integral of this equation one forms the Hamilton principal function $\mathscr{A}(\boldsymbol{Q}, \boldsymbol{q}_0; t)$, which is a special complete integral of (1.1). The trajectories are to be constructed from $\mathscr{A}$, solving the equations $\boldsymbol{p}_0 = \partial\mathscr{A}/\partial\boldsymbol{q}_0$ with respect to Q, and substituting the result into $\boldsymbol{P} = \partial\mathscr{A}/\partial\boldsymbol{Q}$. The Hamilton equations of motion may be considered as those determining the characteristic curves in the phase space. If for each $\boldsymbol{Q}$, $\boldsymbol{q}_0$ and t one knows a trajectory $\boldsymbol{q}(\tau)$, $\boldsymbol{p}(\tau)$ which satisfies $\boldsymbol{q}(t) = \boldsymbol{Q}$, $\boldsymbol{q}(0) = \boldsymbol{q}_0$, the principal function is determined from the action integral

$$\mathscr{A}(\boldsymbol{Q}, \boldsymbol{q}_0; t) = \int_0^t [\boldsymbol{p}(\tau)\dot{\boldsymbol{q}}(\tau) - H(\boldsymbol{q}(\tau), \boldsymbol{p}(\tau); \tau)]\, \mathrm{d}\tau. \tag{1.2}$$

In general, the dynamical equations with the two-point boundary conditions have several solutions, so the action (1.2) is a multivalued function. The action functional defined by means of the integral (1.2) on a set of phase-space trajectories $\boldsymbol{q}(\tau)$, $\boldsymbol{p}(\tau)$ with ends on the hyperplanes $\boldsymbol{q} = \boldsymbol{q}_0$ and $\boldsymbol{q} = \boldsymbol{Q}$ is also a basis of the variational principle. One may interpret the Hamilton equations as the Lagrange–Euler equations for this

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functional (the Livens theorem, presented by Pars 1964, ch. 26; this aspect was also discussed by Ray 1973).

The theory may also be formulated based on the momentum characteristic function, which is related to the coordinate function $\mathscr{A}(\boldsymbol{Q}, \boldsymbol{q}_0; t)$ by means of the double Legendre transformation in both the coordinates $\boldsymbol{Q}$ and $\boldsymbol{q}_0$ (Synge 1960, § 79). Moreover, the dynamics are invariant under the general canonical transformations of the phase space. The invariance is quite manifest in terms of the differential geometry (Woodhouse 1975, Vinogradov and Kuperschmidt 1977),; however, it is hidden in the usual Hamilton–Jacobi formalism. Due to the boundary conditions, the principal function is a non-invariant solution of (1.1).

We present here a modification of the Hamilton–Jacobi theory, which is invariant under linear canonical transformation. In particular, the coordinate–momentum symmetry is obvious. The considered analogue of the Hamilton principal function, *the phase action*, is a function on the phase space and is related to $\mathscr{A}$ by means of the Legendre transformation. The phase action is the solution of the Cauchy problem, so it is unambiguous and this fact is perhaps an advantage over the conventional theory. The phase action is intimately related to the Weyl formulation of quantum mechanics and this is the reason why it may be useful for the semiclassical approach. Besides, the phase action exists for any Hamiltonian, even if there is no Lagrangian (and no $\mathscr{A}$). An important example is the dynamics with anticommuting phase–space variables (Berezin and Marinov 1977).

The present method is developed for a linear phase space. The geometry is not so simple if the configuration space is a non-trivial manifold. Starting from the Lagrangian approach, one gets a cotangent bundle phase space over the configuration manifold. In general Hamiltonian dynamics the phase space may be a manifold or even more sophisticated geometry. For instance, there are situations when the fundamental Poisson brackets (1.4) are not constants. Such theories are of special interest in view of the very important class of system described by singular Lagrangians, involving the Dirac brackets (Dirac 1964, Hanson *et al* 1976). To abstract from the corresponding complications, we suppose that the dynamical system is embedded in $\mathbb{R}^n$ and do not consider the intrinsic formulation at present.

In § 2, the phase action is defined by means of the basic equation (2.3), and its general properties are deduced. Some specific soluble examples are considered in the next two sections; the oscillator and other systems with quadratic Hamiltonians in § 3, free motion on spherical surfaces in arbitrary dimensions in § 4. Perturbative expansion of the phase action is the object of § 5. The role of the phase action in quantum mechanics in view of the semiclassical methods is briefly discussed in § 6. A more general construction of dynamics is presented in the appendix. This extension of classical mechanics may be instructive, even though the corresponding theory is not canonical.

The notations are as follows: $\boldsymbol{q} = \{q_\alpha\}$, $\alpha = 1, \ldots, f$ is the coordinate, $\boldsymbol{p} = \{p_\alpha\}$ is the momentum, f is the number of degrees of freedom, $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p}) = \{x_k\}$, $k = 1, \ldots, n$ is the vector in the phase space, $n = 2f$. The Poisson brackets for any two dynamical variables $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ (functions on the phase space) are written as

$$\{f, g\} = -(\partial f/\partial x_l)\omega_{lm}(\partial g/\partial x_m), \tag{1.3}$$

where $\omega_{lm} = -\omega_{ml}$ and summation over the repeated indices is implied. In particular,

$$\{x_l, x_m\} = -\omega_{lm}, \qquad \{p_\alpha, q_\beta\} = -\{q_\alpha, p_\beta\} = \delta_{\alpha\beta}, \qquad \{q_\alpha, q_\beta\} = \{p_\alpha, p_\beta\} = 0. \tag{1.4}$$

We also use the notations

$$\begin{aligned} &\partial^k = \partial/\partial x_k, \qquad \nabla_k = \omega_{kl}\partial^l, \\ &\tilde{\omega} = -\omega^{-1}, \qquad \tilde{\omega}^{km}\omega_{lm} = \delta^k_l, \\ &(\boldsymbol{x} \circ \boldsymbol{x}') \equiv \tilde{\omega}^{kl} x_k x'_l = \boldsymbol{q} \cdot \boldsymbol{p}' - \boldsymbol{p} \cdot \boldsymbol{q}' = -(\boldsymbol{x}' \circ \boldsymbol{x}) \end{aligned} \tag{1.5}$$

so the Poisson brackets are

$$\{f, g\} = -\partial^k f \, \nabla_k g = \nabla_k f \, \partial^k g = -(\nabla f \circ \nabla g). \tag{1.6}$$

The phase-space trajectory $\boldsymbol{x}(t)$ is the solution of the Hamilton equations

$$\dot{\boldsymbol{x}} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}t = \{H, \boldsymbol{x}\} = \nabla H. \tag{1.7}$$

The time parameter is t or τ.

2. The phase action: general properties

2.1. The definition

The trajectory, starting at $\boldsymbol{\xi}$, i.e. the integral of (1.7) with an initial condition $\boldsymbol{x}(0) = \boldsymbol{\xi}$ may be represented in an implicit form

$$x_l(t) = \xi_l + \omega_{lm} \, \partial\Phi(\boldsymbol{z}; t)/\partial \boldsymbol{z}_m, \qquad \boldsymbol{z} = (\boldsymbol{x} + \boldsymbol{\xi})/2. \tag{2.1}$$

The function $\Phi(\boldsymbol{z}; t)$, introduced in this way, is the basic concept of this work, *the phase action*. To find the trajectory depending on $\boldsymbol{\xi}$ and t, supposing that Φ is known, one has to solve the algebraic equations (2.1) with respect to $\boldsymbol{x}$. It is easily seen that the transformation $\boldsymbol{\xi} \to \boldsymbol{x}$, given by (2.1), preserves the Poisson brackets for any Φ. (In general, writing an automorphism of the phase space implicitly, like (2.1), $\boldsymbol{x} = \boldsymbol{\xi} + \nabla F$ by means of a generating function $F(\boldsymbol{x}, \boldsymbol{\xi})$, one may prove that the automorphism is canonical, provided that $\partial F/\partial \boldsymbol{x} = \partial F/\partial \boldsymbol{\xi}$). In proving the invariance of the Poisson brackets, $\{x_k, x_l\} = \{\xi_k, \xi_l\}$, one has to consider the resolvent matrix $R^l_k = \partial x_k/\partial \xi_l$, which is related through (2.1) to the second derivative of the phase action. Defining the matrix $B^l_k = \omega_{km} \, \partial^2\Phi/\partial z_l \, \partial z_m$, one may show that

$$\begin{aligned} &R = (1 + \tfrac{1}{2}B)(1 - \tfrac{1}{2}B)^{-1} \\ &\omega_{kl}B^l_m = -B^l_k\omega_{lm}, \qquad \omega_{kl}R^l_m = (R^{-1})^l_k\omega_{lm}. \end{aligned} \tag{2.2}$$

The Hamilton equations (1.7) are satisfied, if the function $\Phi(\boldsymbol{z}; t)$ solves a partial differential equation with the zero initial condition

$$\partial\Phi/\partial t = H(\boldsymbol{z} + \tfrac{1}{2}\nabla\Phi; t), \qquad \Phi(\boldsymbol{z}; 0) = 0. \tag{2.3}$$

So we have the standard Cauchy problem for the phase action, contrary to the Hamilton action. Its solution is unique, the fact is established in the theorem by S Kowalewskaya (see in Smirnov 1957).

To learn the relation to the conventional approach, separate coordinates and momenta in equations (2.1) and write $\Phi(\boldsymbol{z}; t) \equiv \Phi(\boldsymbol{r}, \boldsymbol{k}; t)$. Comparing (2.1) with the equations expressing the phase-space trajectory in terms of derivatives of the Hamilton action, one is able to show that $\mathscr{A}$ is just a Legendre transform of Φ:

$$\mathscr{A}(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{\rho}, \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{\rho}; t) = \boldsymbol{k}\boldsymbol{\rho} - \Phi(\boldsymbol{r}, \boldsymbol{k}; t), \qquad \boldsymbol{\rho} = \partial\Phi/\partial \boldsymbol{k}. \tag{2.4}$$

For the inverse transformation

$$\boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{Q} + \boldsymbol{q}_0), \qquad \boldsymbol{\rho} = \boldsymbol{Q} - \boldsymbol{q}_0,$$
$$\boldsymbol{k} = \partial \mathscr{A}/\partial \boldsymbol{\rho} \equiv \tfrac{1}{2}(\partial \mathscr{A}/\partial \boldsymbol{Q} - \partial \mathscr{A}/\partial \boldsymbol{q}_0) = \tfrac{1}{2}(\boldsymbol{P} + \boldsymbol{p}_0).$$

The Legendre transformation is an evident reason for bifurcations, and from a unique Φ one gets a multi-valued $\mathscr{A}$. Equation (2.4) is in close analogy with the usual relation between the Hamiltonian and the Lagrangian, $\mathscr{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - H(\boldsymbol{q}, \boldsymbol{p})$, $\dot{\boldsymbol{q}} = \partial H/\partial \boldsymbol{p}$. The action $\mathscr{A}$ is the integral of $\mathscr{L}$ along the trajectory, the integral for Φ is given by (2.12).

In the following we assume that H is independent of t.

2.2. *Phase action at small times*

The phase action has a regular expansion in powers of t:

$$\Phi(\boldsymbol{z}; t) = \sum_{N=1}^{\infty} \Phi_N(\boldsymbol{z}) t^N / N!. \tag{2.5}$$

Recursive relations for the coefficient functions $\Phi_N(z)$ may be obtained from the Taylor expansion of the right-hand side of (2.3):

$$\partial \Phi/\partial t = H(\boldsymbol{z}) = \sum_{M=1}^{\infty} (2^M M!)^{-1} H^{l_1 \ldots l_M}(z) \prod_{\mu=1}^{M} (\nabla_{l_\mu} \Phi), \tag{2.6}$$

where

$$H^{l_1 \ldots l_M}(\boldsymbol{z}) = \partial M H / \partial z_{l_1} \ldots z_{l_M}.$$

The recursion is

$$\begin{aligned} \Phi_{N+1} = {} & \tfrac{1}{2} H^l \nabla_l \Phi_N + (2^2 2!)^{-1} H^{lm} \sum_{\lambda+\mu=N} C_N^{\lambda\mu} \nabla_l \Phi_\lambda \nabla_m \Phi_\mu \\ & + (2^3 3!)^{-1} H^{lmn} \sum_{\lambda+\mu+\nu=N} C_N^{\lambda\mu\nu} \nabla_l \Phi_\lambda \nabla_m \Phi_\mu \nabla_n \Phi_\nu + \ldots \\ & + 2^{-N} H^{l_1 \ldots l_N} \prod_{\nu=1}^{N} \nabla_{l_\nu} \Phi_1, \end{aligned} \tag{2.7}$$

where $C_N^{\nu_1 \ldots \nu_k} = N!/\nu_1! \ldots \nu_k!$, $C_N^{\lambda\mu}$ is the usual binomial coefficient. Thus any Φ_N is a sum of a number of terms that are products of N derivatives of the Hamiltonian H. The first terms in (2.5) are given by

$$\begin{aligned} & \Phi_1 = H, \qquad \Phi_2 = H^l H_l = -H_l H^l = 0, \qquad \Phi_3 + \tfrac{1}{4} H^{lm} H_l H_m, \qquad \Phi_4 \equiv 0, \\ & \Phi_5 = \tfrac{1}{16}(8 H_k H^{kl} H_{lm} H^{mn} H_n + 4 H_k H^{kl} \omega_{lm} H^{mnp} H_n H_p + H^{klmn} H_k H_l H_m H_n), \end{aligned} \tag{2.8}$$

where $H_l = \nabla_l H$; Φ_2 and Φ_4 vanish because of the antisymmetry of ω_{lm}. In calculating higher terms a graphical technique may be useful. Each term in Φ_N is pictured by a tree graph with $N-1$ lines and N vertices, a vertex representing a derivative $\partial^M H/\partial \boldsymbol{z}^M$ and having M legs. The lines on the graph represent the matrix ω and are oriented. The same tree graphs arise in perturbation theory (§ 5); they are like the Feynman graphs.

2.3. *Time inversion*

It is seen from (2.1) that for fixed t

$$\boldsymbol{X}(0) = \boldsymbol{z} - \tfrac{1}{2} \nabla \Phi, \qquad \boldsymbol{X}(t) = \boldsymbol{z} + \tfrac{1}{2} \nabla \Phi. \tag{2.9}$$

As the Hamiltonian is constant along the trajectory (for conservative systems), equation (2.3) is invariant under the substitution $t \to -t$, $\Phi \to -\Phi$, therefore

$$\Phi(\boldsymbol{z}; t) = -\Phi(\boldsymbol{z}; -t), \qquad \Phi_{2M}(\boldsymbol{z}) = 0. \tag{2.10}$$

Consider a reflection of the phase space, $\boldsymbol{x} \to \sigma\boldsymbol{x}$, where σ is a matrix, anticommuting with ω, and $\sigma^2 = 1$. An example of such reflection is time inversion $\boldsymbol{q} \to \boldsymbol{q}$, $\boldsymbol{p} \to -\boldsymbol{p}$. If the Hamiltonian is invariant under the reflection, one may write, using (2.10),

$$H(\sigma\boldsymbol{x}) = H(x), \qquad \Phi(\sigma\boldsymbol{z}; t) = -\Phi(\boldsymbol{z}; -t) = \Phi(\boldsymbol{z}; t).$$

Under this condition not only $\Phi_{2M} = 0$, but any graph with an odd number of lines vanishes identically. In particular, this relation is applicable to systems symmetrical with respect to time inversion.

2.4. Integral and the variational principle

The solution of the basic equation (2.3) may be written as an integral along the trajectory

$$\Phi(\boldsymbol{z}; t) = \int_0^t [H(\boldsymbol{X}) + 2(\boldsymbol{Z} \circ \dot{\boldsymbol{Z}}) - 2(\boldsymbol{X} \circ \dot{\boldsymbol{Z}})]\, \mathrm{d}\tau \tag{2.11}$$

where $\boldsymbol{X}(\tau)$ is the trajectory, satisfying the Hamiltonian equations, while $\boldsymbol{Z}(\tau) = \frac{1}{2}(\boldsymbol{X}(\tau) + \boldsymbol{X}(0))$ and $\boldsymbol{Z}(t) = \boldsymbol{z}$; the notation (1.5) is used. This representation is especially clear within a more general problem, considered in the appendix. Excluding $\boldsymbol{Z}(\tau)$ one may also write

$$\Phi(\boldsymbol{z}; t) = \int_0^t [H(\boldsymbol{X}) - \tfrac{1}{2}(\boldsymbol{X} \circ \dot{\boldsymbol{X}})]\, \mathrm{d}\tau + (\boldsymbol{\xi} \circ \boldsymbol{z}) \tag{2.12}$$

where $\boldsymbol{\xi} = \boldsymbol{X}(0) = \boldsymbol{Z}(0)$ and the trajectory $\boldsymbol{X}(\tau)$ is specified by the boundary condition $\boldsymbol{X}(t) + \boldsymbol{X}(0) = 2\boldsymbol{z}$. Evidently, (2.12) may be deduced from the usual action integral (1.2), integrating the term $\frac{1}{2}(\boldsymbol{p} \cdot \dot{\boldsymbol{q}})$ by parts and basing it on the relation (2.4).

One may consider (2.11) as the definition of a functional, depending on two trajectories, $\boldsymbol{X}(\tau)$ and $\boldsymbol{Z}(\tau)$, with a single restriction $Z(t) = \boldsymbol{z}$. Then the Euler–Lagrange equations provide with the Hamilton dynamics, supplemented by the relation $\dot{\boldsymbol{X}} = 2\dot{\boldsymbol{Z}}$ and the initial condition $\boldsymbol{X}(0) = \boldsymbol{Z}(0)$. The second variation of the functional (2.11) is

$$\delta^2\Phi = \int_0^t [2(\delta\boldsymbol{Z} \circ \delta\dot{\boldsymbol{Z}}) - 2(\delta\boldsymbol{X} \circ \delta\dot{\boldsymbol{Z}}) + \tfrac{1}{2}\partial^2 H(\boldsymbol{X})/\partial x_k\, \partial x_l\, \delta X_k\, \delta X_l]\, \mathrm{d}\tau. \tag{2.13}$$

Apparently it has no definite sign, so the equations of motion correspond to a saddle point in functional space. In this respect the situation is the same as in the case of the usual action functional (1.2), which has neither minimum, nor maximum on the real phase-space trajectory†.

2.5. Continuous group property

The Lie group property of the Hamiltonian action is expressed by the relation

$$\mathscr{A}(\boldsymbol{q}_2, \boldsymbol{q}_0; t_1 + t_2) = \mathscr{A}(\boldsymbol{q}_2, \boldsymbol{q}_1; t_2) + \mathscr{A}(\boldsymbol{q}_1, \boldsymbol{q}_0; t_1), \tag{2.14}$$

† The remark is due to Hilbert, see Pars 1964, § 26.3.

where the intermediate coordinate $\boldsymbol{q}_1$ is to be excluded from the condition

$$\partial\mathscr{A}(\boldsymbol{q}_2, \boldsymbol{q}_1; t_2)/\partial\boldsymbol{q}_1 + \partial\mathscr{A}(\boldsymbol{q}_1, \boldsymbol{q}_0; t_1)/\partial\boldsymbol{q}_1 = 0, \tag{2.15}$$

which means that the momentum is continuous at $\tau = t_1$. The corresponding property of the phase action is written as

$$\Phi(\boldsymbol{z}; t_1 + t_2) = \Phi(\boldsymbol{z}_1; t_1) + \Phi(\boldsymbol{z}_2; t_2) + 4S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}) \tag{2.16}$$

where, with the notation (1.5),

$$S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}) = \tfrac{1}{2}[(\boldsymbol{z}_1 \circ \boldsymbol{z}_2) + (\boldsymbol{z}_2 \circ \boldsymbol{z}) + (\boldsymbol{z} \circ \boldsymbol{z}_1)] \tag{2.17}$$

and the phase-space vectors $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are to be excluded from the relations

$$\boldsymbol{z}_1 = \boldsymbol{z} - \tfrac{1}{2}\nabla\Phi(\boldsymbol{z}_2; t_2), \qquad \boldsymbol{z}_2 = \boldsymbol{z} + \tfrac{1}{2}\nabla\Phi(\boldsymbol{z}_1; t_1). \tag{2.18}$$

The clearest way to deduce (2.16) and (2.18) is to consider the geometry on the phase plane for one degree of freedom (figure 1). Generalisation to any $f > 1$ is

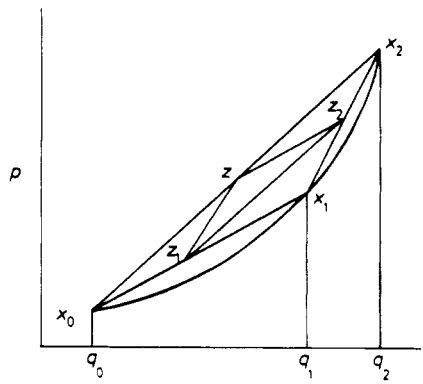

Figure 1. Trajectory on the phase plane. An interpretation of the Lie group property of the phase action, given by (2.16). The point z is in the middle of the segment x_0x_2.

straightforward. The term $\boldsymbol{k}(\boldsymbol{q}_1 - \boldsymbol{q}_0)$ in the relation (2.4) is the area of the rectangular trapezium $(\boldsymbol{q}_0, \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{q}_1)$. Therefore, the third term in (2.16) must be equal to the area of the triangle $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2)$, or to the area of $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z})$, multiplied by 4. The area of $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z})$ is given by (2.17), because $(\boldsymbol{x} \cdot \boldsymbol{y})/2$, defined in (1.5), is just the $(\pm)$ area of the triangle $(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{y})$; plus for counter-clockwise orientation, minus for the clockwise orientation. It is seen from figure 1 that $\boldsymbol{z} - \boldsymbol{z}_1 = \frac{1}{2}(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, $\boldsymbol{z}_2 - \boldsymbol{z} = \frac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x}_0)$. Using equation (2.1), one gets (2.18). On the other hand, these equations are analogous to (2.15), stating that $\Phi(\boldsymbol{z}; t_1 + t_2)$ is stationary against variations of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$,

$$\partial(\Phi(\boldsymbol{z}_1) + 4S)/\partial\boldsymbol{z}_1 = 0, \qquad \partial(\Phi(\boldsymbol{z}_2) + 4S)/\partial\boldsymbol{z}_2 = 0. \tag{2.19}$$

One may verify directly that the right-hand side of (2.16) depends only on the sum $t_1 + t_2$. One may also get (2.16), solving equation (2.3) with a non-zero initial condition, $\Phi(\boldsymbol{z}; 0) = \Phi_1(\boldsymbol{z})$, and using the integral representation (2.12).

2.6. Energy representation

If the Hamiltonian H is independent of t, the Legendre transformation is useful once

more, just as in the conventional formalism. Define the *energy phase action*, in analogy to the 'truncated' action in the Jacobi theory,

$$\Omega(\boldsymbol{z}; E) = Et - \Phi(\boldsymbol{z}; t), \qquad E = \partial\Phi/\partial t. \tag{2.20}$$

It satisfies the partial differential equation

$$H(\boldsymbol{z} - \tfrac{1}{2}\nabla\Omega) = E, \tag{2.21}$$

and vanishes with its E-derivative on the 'energy shell'

$$\Omega(\boldsymbol{z}; H(\boldsymbol{z})) = 0 = \partial\Omega/\partial E, \qquad E = H(\boldsymbol{z}). \tag{2.22}$$

The transformation (2.20) is, in general, not unique, and $\Omega(\boldsymbol{z}; E)$ is multi-valued, as in the Hamilton–Jacobi theory. It also has a quite similar geometric interpretation. Using (2.12), write the integral representation

$$\Omega(\boldsymbol{z}; E) = \tfrac{1}{2}\int (\boldsymbol{X} \circ \mathrm{d}\boldsymbol{X}) - \tfrac{1}{2}(\boldsymbol{x}_0 \circ \boldsymbol{x}_1) \tag{2.23}$$

where the integration is along the trajectory lying on the energy shell $H(\boldsymbol{x}) = E$, $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are the initial point and the final point on the trajectory, $z = \frac{1}{2}(\boldsymbol{x}_0 + \boldsymbol{x}_1)$. In the case of one degree of freedom, Ω is the area of the segment limited by the trajectory and its chord (see figure 1).

It is shown in § 5 that if the system consists of two systems with no interaction

$$H(\boldsymbol{x}) = H_1(\boldsymbol{x}) + H_2(\boldsymbol{x}), \qquad \{H_1, H_2\} = 0, \tag{2.24}$$

the total phase action may be obtained from the partial phase actions for the subsystems,

$$\begin{aligned} &\Phi(\boldsymbol{z}; t) = \Phi_1(\boldsymbol{z}_1; t) + \Phi_2(\boldsymbol{z}_2; t) - 4S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}) \\ &\boldsymbol{z} = \boldsymbol{z}_1 - \tfrac{1}{2}\nabla\Phi_2 = \boldsymbol{z}_2 + \tfrac{1}{2}\nabla\Phi_1. \end{aligned} \tag{2.25}$$

Using relations (5.4), from (2.25) a 'superposition principle' for the energy phase action, one gets

$$\Omega(\boldsymbol{z}; E_1 + E_2) = \Omega_1(\boldsymbol{z}_1; E_1) + \Omega_2(\boldsymbol{z}_2; E_2) + 4S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}), \tag{2.26}$$

where

$$H_1(\boldsymbol{z}_1 - \tfrac{1}{2}\nabla\Omega_1) = E_1, \qquad H_2(\boldsymbol{z}_2 - \tfrac{1}{2}\nabla\Omega_2) = E_2, \qquad \boldsymbol{z}_1 + \tfrac{1}{2}\nabla\Omega_2 = \boldsymbol{z}_2 - \tfrac{1}{2}\nabla\Omega_1 = \boldsymbol{z}. \tag{2.27}$$

The analogy with the group property, given by (2.16), is manifest. However, in this case E_1 and E_2 are not arbitrary, and may be found from equations (2.27).

3. Systems with quadratic Hamiltonians

Apply the present formalism to a dynamical system with the Hamiltonian

$$H(\boldsymbol{x}) = \tfrac{1}{2}F^{kl}x_k x_l \equiv -\tfrac{1}{2}(\boldsymbol{x} \circ K\boldsymbol{x}) = \tfrac{1}{2}(K\boldsymbol{x} \circ \boldsymbol{x}), \qquad K^l_{\ m} = \omega_{mn}F^{nl}, \tag{3.1}$$

where F^{kl} are elements of a symmetrical 'frequency matrix', and we use the notation (1.5). Note that, because of the identity $F^{\mathrm{T}} = F$, the matrix K anticommutes with $\tilde{\omega}$, i.e. $\tilde{\omega}^{ln}K^m_{\ n} = -K^l_{\ n}\tilde{\omega}^{nm}$. By means of a canonical linear transformation the Hamiltonian may be reduced to a standard form. The general classification of these forms was

elaborated by Williamson (1937), the results may be found also in the book by Arnold (1974, addendum 6). If F is diagonal and has positive eigenvalues, the system is an (anisotropic) oscillator; if some of the eigenvalues are zero, there is free motion in the corresponding coordinates.

The equations of motion are linear

$$\dot{\boldsymbol{x}} = K\boldsymbol{x} \tag{3.2}$$

and their solution is written as follows

$$\boldsymbol{X}(t) = = R(t)\boldsymbol{X}(0), \qquad R(t) = \exp(tK). \tag{3.3}$$

Note that $\det R = 1$ because $\operatorname{Tr} K = \omega_{lm}F^{ml} = 0$. Evidently, the phase action is also quadratic,

$$\Phi(\boldsymbol{z}; t) = \tfrac{1}{2}(\boldsymbol{B}(t)\boldsymbol{z} \circ \boldsymbol{z}). \tag{3.4}$$

In view of equation (2.3), the matrix $B(t)$ satisfies the Riccati equation and may be related to the resolvent matrix $R(t)$,

$$\dot{B} = (1 - \tfrac{1}{2}B)K(1 + \tfrac{1}{2}B), \qquad B(0) = 0, \qquad B = 2(R-1)(R+1)^{-1}. \tag{3.5}$$

Three particular applications of this general result are mfpecial interest.

3.1. Isotropic oscillator

The frequency matrix is unity up to a factor, while $\omega^2 = -1$; that is true for the usual $(\boldsymbol{q}, \boldsymbol{p})$ representation of the phase space. Now

$$\begin{aligned} &F^{kl} = \kappa\delta_{kl}, \qquad R^l_k(t) = \cos\kappa t + \omega_{kl}\sin\kappa t, \\ &B^l_k(t) = 2\omega_{kl}\tan(\kappa t/2), \qquad \Phi(\boldsymbol{z}; t) = z^2\tan(\kappa t/2), \end{aligned} \tag{3.6}$$

where κ is the frequency.

3.2. Free motion

$$\boldsymbol{H}(\boldsymbol{x}) = \tfrac{1}{2}(M^{-1})_{\alpha\beta}p_\alpha p_\beta, \qquad K = \begin{pmatrix} 0 & M^{-1} \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

where M is the 'mass matrix'; $\alpha, \beta = 1, \ldots, f$. In this case†

$$K^2 = 0, \qquad R(t) = 1 + tK, \qquad B(t) = tK, \qquad \Phi(\boldsymbol{z}; t) = tH(\boldsymbol{z}). \tag{3.8}$$

3.3. Rotation

The rotation of the configuration space may be considered as the time evolution, induced by the Hamiltonian

$$H(\boldsymbol{x}) = \Theta^{\alpha\beta}L_{\alpha\beta}(\boldsymbol{x}), \qquad L_{\alpha\beta} = q_\alpha p_\beta - q_\beta p_\alpha, \tag{3.9}$$

where Θ is the angular velocity matrix and $L_{\alpha\beta}$ is the angular momentum. It is assumed

† It is evident directly from equation (2.3) that for any $H(\boldsymbol{x}) = V(\boldsymbol{p})$ the phase action is also independent of $\boldsymbol{q}$: $\Phi = tV(\boldsymbol{p})$. The same is true for $H(\boldsymbol{x}) = V(\boldsymbol{q})$: $\Phi = tV(\boldsymbol{q})$. Note that the Hamilton action does not exist in this trivial case.

that $\Theta = -\Theta^{\mathrm{T}}$, so the rotation matrix $\boldsymbol{D} = \exp(t\Theta)$ is orthogonal. The phase action is

$$\Phi(\boldsymbol{z}; t) = \Gamma^{\alpha\beta}(t) L_{\alpha\beta}(\boldsymbol{z}), \qquad \Gamma = 2(D-1)(D+1)^{-1}. \tag{3.10}$$

To conclude this section we present the solution of a more general problem: the non-conservative system with a variable frequency matrix and under the influence of a homogeneous external field. The Hamiltonian and the phase action are

$$\begin{aligned} &\boldsymbol{H}(\boldsymbol{x}; t) = \tfrac{1}{2}(\boldsymbol{K}(t)\boldsymbol{x} \circ \boldsymbol{x}) + (\boldsymbol{f}(t) \circ \boldsymbol{x}), \\ &\Phi(\boldsymbol{z}; t, t_0) = \tfrac{1}{2}[B(t, t_0)(\boldsymbol{z}+\boldsymbol{b}) \circ (\boldsymbol{z}+\boldsymbol{b})] + 2(\boldsymbol{b} \circ \boldsymbol{z}) + \phi(t, t_0). \end{aligned} \tag{3.11}$$

Formulae (3.5) are valid, while $\phi(t, u)$, the matrix $B(t, t_0)$ and the vector $\boldsymbol{b}(t, t_0)$ are obtained from the equations

$$\begin{aligned} &\dot{R} = K(t)R, \qquad R(t_0, t_0) = 1, \qquad B = 2(R-1)(R+1)^{-1}, \\ &b(t, t_0) = \tfrac{1}{2}\int_{t_0}^{t} R^{-1}(\tau, t_0) \boldsymbol{f}(\tau)\, \mathrm{d}\tau, \\ &\phi(t, t_0) = \int_{t_0}^{t} (\boldsymbol{f}(\tau) \circ R(\tau) b(\tau))\, \mathrm{d}\tau. \end{aligned} \tag{3.12}$$

This problem may be of interest in view of the perturbation theory.

4. The rotator and the Kepler problem

A material point on a spherical surface is usually described by means of the free Lagrangian with the constraint $\boldsymbol{q}^2 = a^2$, where a is the radius of the sphere. Start from another point and consider the Hamiltonian

$$H = \tfrac{1}{4} L_{\alpha\beta} L_{\alpha\beta} \equiv \tfrac{1}{2}\Lambda^2, \tag{4.1}$$

where $L_{\alpha\beta}$ is the angular momentum (3.9) and

$$\Lambda^2 = \boldsymbol{p}^2\boldsymbol{q}^2 = (\boldsymbol{p} \cdot \boldsymbol{q})^2. \tag{4.2}$$

The Legendre transformation to the velocities is ambiguous, because $\det(\partial^2 \boldsymbol{H}/\partial p_\alpha \partial p_\beta) = 0$, and the constraint $(\boldsymbol{q} \cdot \dot{\boldsymbol{q}}) = 0$ appears. The Hamiltonian is in a sense degenerate. A similar situation is investigated by Dirac (a review in Dirac 1964) for the inverse problem, constructing a Hamiltonian from a degenerate Lagrangian. However, we need not go beyond the framework of the phase space approach and do not discuss the Lagrange formulation here.

The equations of motion are

$$\dot{\boldsymbol{q}} = \boldsymbol{q}^2\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{q})\boldsymbol{q}, \qquad \dot{\boldsymbol{p}} = -\boldsymbol{p}^2\boldsymbol{q} + (\boldsymbol{p} \cdot \boldsymbol{q})\boldsymbol{p} \tag{4.3}$$

and they have the solution

$$\begin{aligned} &\boldsymbol{q} = a(\boldsymbol{u} \cos \Lambda t + \boldsymbol{v} \sin \Lambda t) \\ &\boldsymbol{p} = b[\boldsymbol{u} \cos(\Lambda t + \theta) + \boldsymbol{v} \sin(\Lambda t + \theta)]. \end{aligned} \tag{4.4}$$

Here a, b and θ are constants, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are constant orthonormal vectors, so that the

integrals of motion are written as follows:

$$\begin{aligned} &\boldsymbol{q}^2 = a^2, \qquad \boldsymbol{p}^2 = b^2, \qquad \boldsymbol{p}\cdot\boldsymbol{q} = ab\cos\theta, \\ &\Lambda = ab\sin\theta, \qquad L_{\alpha\beta} = \Lambda(u_\alpha v_\beta - u_\beta v_\alpha). \end{aligned} \tag{4.5}$$

As it is suggested by the power expansion (2.5) for the Hamiltonian (4.1), the phase-space dependence of the action is given by a function of Λ,

$$\Phi(\boldsymbol{z}; t) = \phi(\Lambda, t), \qquad \partial\phi/\partial t = \tfrac{1}{2}\Lambda^2[1 + \tfrac{1}{4}(\partial\phi/\partial\Lambda)^2]^2. \tag{4.6}$$

Dimensional analysis shows that the last equation has a solution in terms of a function of a single variable

$$\begin{aligned} &w = \Lambda t, \qquad \phi(\Lambda, t) = \Lambda F(w), \qquad F(0) = 0, \\ &F' = \tfrac{1}{2}[1 + \tfrac{1}{4}(F + wF')^2]^2, \qquad F' = \mathrm{d}F/\mathrm{d}w. \end{aligned} \tag{4.7}$$

Solving the last equation for $F + wF'$ one gets an ordinary differential equation of the Lagrange type (note that at positive w not far from zero, evidently, $F' > 0$ and $F > 0$), which is reducible to quadratures. The resulting solution may be presented in a parametric form

$$w = 2s\cos^2 s, \qquad F' = \tfrac{1}{2}(\cos s)^{-4}, \qquad F = 2\tan s - wF'. \tag{4.8}$$

The function $F(w)$ is regular for $0 \leqslant s < s_1$, where $s_1 \approx 0{\cdot}653$ is the first positive root of the equation

$$1 - 2s\tan s = 0. \tag{4.9}$$

At $s = s_1$ one has $\mathrm{d}w/\mathrm{d}s = 0$ and $\mathrm{d}F/\mathrm{d}s = 0$; this point corresponds to a cusp on the integral curve. In the complex w plane $F(w)$ has branch points at $w = w_n = 8s_n^3(1 + 4s_n^2)^{-1}$, where s_n are roots of the equation (4.9). Note that at imaginary s both w and $F(w)$ are pure imaginary, while F' is real.

The solution of the equations of motion, given by (4.4), may be obtained also from (2.1), so that

$$\boldsymbol{q}(t) = \boldsymbol{q}(0) + \frac{\partial\Phi}{\partial\Lambda}\frac{\partial\Lambda}{\partial\boldsymbol{k}}, \qquad \boldsymbol{p}(t) = \boldsymbol{p}(0) - \frac{\partial\Phi}{\partial\Lambda}\frac{\partial\Lambda}{\partial\boldsymbol{r}},$$

where

$$\begin{aligned} &\boldsymbol{k} = \tfrac{1}{2}(\boldsymbol{p}(t) + \boldsymbol{p}(0)), \qquad \boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{q}(t) + \boldsymbol{q}(0)), \qquad \Lambda = \Lambda(\boldsymbol{r}, \boldsymbol{k}), \\ &\partial\phi/\partial\Lambda = F + wF' = 2\tan s. \end{aligned} \tag{4.10}$$

It may also be seen that $2s = t\Lambda(\boldsymbol{q}, \boldsymbol{p}) = t\Lambda(\boldsymbol{r}, \boldsymbol{k})/\cos^2 s$.

Now consider the Kepler problem. It is known (see in particular Gyorgyi 1968), that the f-dimensional dynamical system with the Hamiltonian

$$H = -\tfrac{1}{2}g^2/\Lambda^2 \tag{4.11}$$

under the supplementary condition $(\boldsymbol{p}\cdot\boldsymbol{q}) = 0$ is equivalent to a combination of negative-energy regions of the phase spaces for the $(f-1)$-dimensional Kepler problems with the Hamiltonians $H_{\mathrm{K}} = \boldsymbol{P}^2/2m - \zeta/Q$ for various masses m and couplings ζ. The constants of motion are given by (4.5). Only the states with $\theta = \frac{1}{2}\pi$ are of interest

for the Kepler problem. The parameters m and ζ are related to the constants of motion for the Hamiltonian (4.11),

$$m = a^2 b^4 / g^2, \qquad \zeta = g^2 / ab^2. \tag{4.12}$$

The f-dimensional momenta $\boldsymbol{p}$ are the Fock variables, $\boldsymbol{q}$ are the canonical conjugated coordinates. The 'physical' momenta $\boldsymbol{P}_\nu$, $\nu = 1, \ldots, f-1$, are related to $\boldsymbol{p}$ by means of the stereographic projection, the relation between $\boldsymbol{Q}$ and $\boldsymbol{q}$ is more complicated. The explicit formulae as well as the references are given by Gyorgyi (1968).

The equations of motion and their solutions are

$$\begin{aligned} &\dot{\boldsymbol{q}} = g^2(\boldsymbol{q}^2 \cdot \boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{q})\boldsymbol{q})/\Lambda^4, \qquad \dot{\boldsymbol{p}} = -g^2(\boldsymbol{p}^2 \cdot \boldsymbol{q} - (\boldsymbol{p} \cdot \boldsymbol{q})\boldsymbol{p})/\Lambda^4, \\ &\boldsymbol{q} = a(\boldsymbol{u} \cos \epsilon t + \boldsymbol{v} \sin \epsilon t), \qquad \epsilon = g^2/\Lambda^3, \\ &\boldsymbol{p} = b[\boldsymbol{u} \cos(\epsilon t + \theta) + \boldsymbol{v} \sin(\epsilon t + \theta)]. \end{aligned} \tag{4.13}$$

Proceeding along the same lines as in the case of the rotator, one gets for the phase action

$$\Phi(\boldsymbol{z}; t) = \Lambda F(w), \qquad w = g^2 t / \Lambda^3, \qquad F' = -\tfrac{1}{2}[1 + (F - 3wF')^2]^{-2}. \tag{4.14}$$

The solution is given by the parametric form

$$w = 2s(\cos s)^{-6}, \qquad F' = -\tfrac{1}{2}\cos^4 s, \qquad F = 2 \tan s + 3wF'. \tag{4.15}$$

In this case $F(w)$ has no singularities on the real axis; w varies from 0 to ∞ and $F(w)$ from 0 to $-\infty$ when $0 \leqslant s < \frac{1}{2}\pi$.

5. Perturbation theory

Consider a Hamiltonian of the form

$$H(\boldsymbol{x}) = H_1(\boldsymbol{x}) + \lambda H_2(\boldsymbol{x}), \tag{5.1}$$

where λ is a small parameter. Our purpose now is to expand the phase action in powers of λ. Having in mind the Lie group structure described in § 2, write

$$\Phi(\boldsymbol{z}; t) = \Phi_1(\boldsymbol{z}_1; t) + \Phi_2(\boldsymbol{z}_2; t) - 4S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}), \tag{5.2}$$

where the bilinear form S is given in (2.17), and the variables $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are functions of $\boldsymbol{z}$ and t, to be found from the equations

$$\boldsymbol{z}_1 = \boldsymbol{z} + \tfrac{1}{2}\nabla\Phi_2(\boldsymbol{z}_2; t), \qquad \boldsymbol{z}_2 = \boldsymbol{z} - \tfrac{1}{2}\nabla\Phi_1(\boldsymbol{z}_1; t). \tag{5.3}$$

With these definitions the following useful relations may be readily obtained:

$$\partial\Phi/\partial t = \partial\Phi_1/\partial t + \partial\Phi_2/\partial t, \tag{5.4}$$

$$\nabla\Phi = \nabla\Phi_1 + \nabla\Phi_2, \tag{5.5}$$

where for each function we mean its partial derivatives with respect to its own arguments. Substitute the expression (5.2) into the basic equation (2.3) and note that in view of equations (5.3)

$$\boldsymbol{z} + \tfrac{1}{2}\nabla\Phi = \boldsymbol{z}_1 + \tfrac{1}{2}\nabla\Phi_1 = \boldsymbol{z}_2 + \tfrac{1}{2}\nabla\Phi_2 + \nabla\Phi_1. \tag{5.6}$$

It is natural to assume now that Φ_1 is the phase action for the unperturbed system and to write

$$\begin{aligned} &\partial\Phi_1(\boldsymbol{z}_1; t)/\partial t = H_1(\boldsymbol{z}_1 + \tfrac{1}{2}\nabla\Phi_1), \\ &\partial\Phi_2(\boldsymbol{z}_2; t)/\partial t = \lambda H_2(\boldsymbol{z}_2 + \tfrac{1}{2}\nabla\Phi_2 + \nabla\Phi_1). \end{aligned} \tag{5.7}$$

The variable $\boldsymbol{z}_1$ in the last equation is to be substituted as a function of $\boldsymbol{z}_2 + \frac{1}{2}\nabla\Phi_2$, which may be found from equation (5.6):

$$\boldsymbol{z}_1 - \tfrac{1}{2}\nabla\Phi_1 = \boldsymbol{z}_2 + \tfrac{1}{2}\nabla\Phi_2 \equiv \boldsymbol{\xi}. \tag{5.8}$$

The solution of this algebraic equation is evidently related to the phase-space trajectory starting at $\boldsymbol{\xi}$ and governed by the Hamiltonian H_1 (cf equation (2.1))

$$\boldsymbol{z}_1 = \tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{\xi}), \qquad \boldsymbol{x} = \boldsymbol{X}(\boldsymbol{\xi}, t), \tag{5.9}$$

$$\partial\boldsymbol{X}/\partial t = \nabla H_1(\boldsymbol{X}), \qquad \boldsymbol{X}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi}. \tag{5.1}$$

The situation is shown in figure 2. Now (5.7) acquires the normal form of the phase action equation with a time-dependent Hamiltonian,

$$\begin{aligned} &\partial\Phi_2(\boldsymbol{z}; t)/\partial t = \lambda G(\boldsymbol{z} + \tfrac{1}{2}\nabla\Phi_2; t), \qquad \Phi_2(\boldsymbol{z}; 0) = 0, \\ &G(\boldsymbol{\xi}; t) \equiv H_2(\boldsymbol{X}(\boldsymbol{\xi}; t)). \end{aligned} \tag{5.11}$$

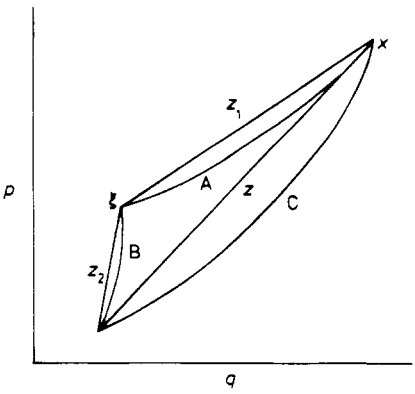

Figure 2. Motion for a Hamiltonian (5.1). Curve C is the phase space trajectory, curve A is a trajectory mastered by the unperturbed Hamiltonian H_1 and ending at the same point, curve B is a trajectory for the time-dependent Hamiltonian (5.11).

If H_2 is an integral of motion for the system with the Hamiltonian H_1, $\{H_1, H_2\} = 0$, then there is no explicit time dependence in equation (5.11), $G(\boldsymbol{\xi}; t) \equiv H_2(\boldsymbol{\xi})$, and its solution is reduced to the standard problem with the Hamiltonian H_2

$$\Phi_2(\boldsymbol{z}; t) = \Psi(\boldsymbol{z}; \lambda t), \qquad \partial\Psi(\boldsymbol{z}; t)/\partial t = H_2(\boldsymbol{z} + \tfrac{1}{2}\nabla\Psi). \tag{5.12}$$

Thus at $\lambda = 1$ we get the result presented in equation (2.25).

It is evident that Φ_2, as well as $\boldsymbol{z}-\boldsymbol{z}_1$ and S in (5.2), is proportional to λ. Using equation (5.11), one may calculate the expansion in powers of λ for Φ_2 (cf (2.6))

$$\Phi_2(\boldsymbol{z};t)=\sum_{N=1}^{\infty}\lambda^N\phi_N(\boldsymbol{z};t)/N!,$$
$$\partial\phi_1(\boldsymbol{z};t)/\partial t=G(\boldsymbol{z};t), \tag{5.13}$$

$$\partial\phi_{N+1}/\partial t=\tfrac{1}{2}G^l\nabla_1\phi_N+(2^2 2!)^{-1}G^{mn}\sum_{\mu+\nu=N}C_N^{\mu\nu}\nabla_m\phi_\mu\nabla_n\phi_\nu$$
$$+\ldots+2^{-N}G^{l_1\ \ l_N}\prod_{\nu=1}^{N}\nabla_{l_\nu}\phi_1, \tag{5.14}$$

where we use the same form as in equation (2.7), and $G^{l_1\ldots l_M}=\partial^M G(\boldsymbol{z};t)/\partial z_{l_1}\ldots\partial z_{l_M}$. The functions ϕ_N are obtained subsequently by means of integration in t. The first terms are

$$\phi_1(\boldsymbol{z}:t)=\int_0^t G(\boldsymbol{z};\tau)\,\mathrm{d}\tau,$$
$$\phi_2(\boldsymbol{z};t)=-\tfrac{1}{2}\int_0^t \mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\{G(\boldsymbol{z};\tau_1),G(\boldsymbol{z};\tau_2)\}, \tag{5.15}$$
$$\phi_3(\boldsymbol{z};t)=\tfrac{1}{4}\int_0^t \mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\Big(\int_0^{\tau_2}\mathrm{d}\tau_3\{G(\tau_1),\{G(\tau_2),G(\tau_3)\}\}$$
$$+\int_0^{\tau_1}\mathrm{d}\tau_3 G^{kl}(\tau_1)\nabla_k G(\tau_2)\nabla_1 G(\tau_3)\Big).$$

Just as in § 2.2, one may use a graphical technique. Any ϕ_N is a sum of terms represented by tree graphs with $N-1$ lines and N vertices. However, now any vertex depends on its own time, and one has to calculate the multiple integral in the times, having in mind a time ordering. If the unperturbed Hamiltonian H_1 is quadratic in $\boldsymbol{x}$ (see § 3), the graphs are in close analogy to the Feynman graphs. The time dependence of G is due to the presence of the matrix $R(t)$, given in (3.3), and we have

$$G(\boldsymbol{\xi};t)=H_2(R(t)\boldsymbol{\xi}),\qquad \partial G/\partial\xi_1=R_k^l(t)\,\partial H_2/\partial x_k. \tag{5.16}$$

Each leg of a vertex contributes a matrix factor $\boldsymbol{R}(\tau)$. One may redefine the graphical notations and assume that the vertices represent derivatives of the perturbing Hamiltonian, $\partial^M H_2/\partial\boldsymbol{x}\ldots\partial\boldsymbol{x}$, while the lines are associated with the propagator

$$D_{kl}(\tau_1-\tau_2)=R_k^m(\tau_1)\omega_{mn}R_l^n(\tau_2)=R_k^m(\tau_1-\tau_2)\omega_{ml}. \tag{5.17}$$

We have used here the general property $\omega\boldsymbol{R}=\boldsymbol{R}^{-1}\omega$.

A defect of the expansion (5.13) is that its convergence may depend on t. An example is a system containing non-interacting subsystems, when (5.12) is valid. In this case the expansion parameter in (5.13) is in fact λt; the expansion coincides with (2.5), and $\phi_N(\boldsymbol{z};t)=t^N\Phi_N(\boldsymbol{z})$. A way to improve the convergence is to eliminate from H_2 a function, constant along the trajectory $\boldsymbol{X}(\boldsymbol{\xi};t)$, governed by H_1. Instead of (5.1) one may write

$$H=(H_1+\lambda K)+\lambda(H_2-K)=H_1'+\lambda H_2',\qquad \{H_1,K\}=0, \tag{5.18}$$

choosing $K(\boldsymbol{x})$ in such a way as to make the time average of H_2', which is just ϕ_1 in

(5.15), as small as possible at large t. As to the phase action for the new unperturbed Hamiltonian H_1', one has to solve the basic equation (2.3) with the Hamiltonian $K(\boldsymbol{x})$ and then employ equation (2.25). Another way to modify the perturbative expansion of the phase action is to use repeatedly the generalised sum, defined in (5.2), instead of writing the series (5.13) directly. Future investigations will show whether such modifications are effective. To conclude this section, note that the expansion of the usual Hamilton action is a rather complicated procedure, which avoids the invariant formulation (see e.g. Sen Gupta 1967).

6. Relation to quantum theory

The phase action is of interest for quantum theory, because it gives the phase-space symbol of the Green operator in the semiclassical approximation. It is known that semiclassical methods are very fruitful in quantum theory (see for example an excellent review by Berry and Mount 1972).

Formulation of quantum mechanics in terms of the phase-space symbols, invented by Weyl (1927) (see also Wigner 1932), is quite appropriate for the semiclassical approach, because the classical dynamical variable, corresponding to a quantal operator, is just the limit $\hbar \to 0$ of its symbol. The symbols are related to matrix elements of the operators by means of the Fourier transformation. In particular, one may write the kernel of the time-dependent Green function,

$$\langle \boldsymbol{q}_2 | \exp(-\mathrm{i}\hat{H}t/\hbar) | \boldsymbol{q}_1 \rangle = (2\pi\hbar)^{-f} \int G(\tfrac{1}{2}\boldsymbol{q}_1 + \tfrac{1}{2}\boldsymbol{q}_2, \boldsymbol{p}; t) \exp(-\mathrm{i}\boldsymbol{p} \cdot (\boldsymbol{q}_1 - \boldsymbol{q}_2)/\hbar)\, \mathrm{d}^f\boldsymbol{p}, \tag{6.1}$$

where the symbol is

$$G(\boldsymbol{x}; t) = A(\boldsymbol{x}; t) \exp(-\mathrm{i}\phi(\boldsymbol{x}; t)/\hbar).$$

A and ϕ are real functions. It is quite adequate to consider the phase $\phi(\boldsymbol{x}; t)$ as the symbol of Schwinger's quantum action. If ϕ is known, the amplitude A is determined from the unitarity of Green's operator. It may be shown that in the semiclassical approximation the function ϕ is just the phase action,

$$\begin{aligned} &\phi(\boldsymbol{x}; t) = \Phi(\boldsymbol{x}; t) + \mathrm{O}(\hbar^2), \\ &A(\boldsymbol{x}; t) = [\det(1 - B^2/4)]^{1/4} + \mathrm{O}(\hbar^2). \end{aligned} \tag{6.2}$$

The matrix B is given by

$$B_k^l(\boldsymbol{x}; t) = \nabla_k\, \partial^l \Phi(\boldsymbol{x}; t). \tag{6.3}$$

For the case of a quadratic Hamiltonian (§ 3) equations (6.2) are *exact*; quantal corrections do not arise; the matrix B does not depend on $\boldsymbol{x}$ and is presented in (3.4) (see also (3.12) for the time-dependent case).

The equation (2.3) was first written by Berezin and Shubin (1972) in the context of the semiclassical approximation. The starting point was an integro-differential equation for the symbol $G(\boldsymbol{x}; t)$ representing the operator equation

$$\mathrm{i}\hbar\, \partial\hat{G}/\partial t = \hat{H}\hat{G}. \tag{6.4}$$

In the limit $\hbar \to 0$, the integral expressing the operator multiplication in the right-hand

side is calculated by means of the stationary phase method and the result is two real first-order partial differential equations. One of them, for the phase, is the equation (2.3), the other, for the amplitude, is linear and may be integrated in closed form (6.2). To obtain the usual Green function one should substitute the result (6.2) into (6.1) and apply the stationary phase method once more. The Fourier integral is then reduced to the Legendre transformation for the exponent, providing the mapping (2.4). Thus the sum over the stationary phase points in the phase space is converted to the sum over classical trajectories in the coordinate space. The terms are of the form $\exp[\mathrm{i}\mathscr{A}(\boldsymbol{q}_1, \boldsymbol{q}_2; t)/h - \mathrm{i}\gamma\pi/2]$, where $\mathscr{A}$ is the Hamilton action and γ is interpreted as the number of caustics intersected by the trajectory. A presentation of the multi-dimensional stationary phase method and the corresponding relation between the Fourier transformation and the Legendre transformation are given in a book by Maslov and Fedoryuk (1976). Note that contrary to the Hamilton action $\mathscr{A}$ the phase action Φ is unique for fixed $\boldsymbol{z}$; and there are no caustics in the phase space, they appear when the trajectory is projected on to the coordinate space. Probably, it is not necessary to follow this line, involving new errors of order $\hbar^2$ in approximate calculation of the integral (6.1). In some applications the symbol is as good as the Green function.

It is remarkable that some formulae of this paper are clearer in view of their quantal counterparts. For instance, equation (2.3) is related to (6.4), equation (2.16) is just the identity $\hat{G}(t_1+t_2) = \hat{G}(t_2)\hat{G}(t_1)$, equality (2.25) is equivalent to

$$\exp[-\mathrm{i}(\hat{H}_1+\hat{H}_2)t/\hbar] = \exp(-\mathrm{i}\hat{H}_1 t/\hbar)\exp(-\mathrm{i}\hat{H}_2 t/\hbar), \qquad \text{if } [\hat{H}_1, \hat{H}_2] = 0. \tag{6.5}$$

The appearance of the term $S(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z})$ in (2.16), (2.25), (2.26) and (5.2) is due to the multiplication law for the Weyl symbols, while (2.18) or (2.19) arise from the stationary phase approximation. The fundamental equation of the perturbation theory (5.11) has its prototype in Dirac's interaction picture.

A concise review of the formalism of symbols, together with the relevant references, may be found in the appendix to a paper by Berezin and Marinov (1977) (we use the same notations here). In that work an invariant form of the phase-space path integral is also presented:

$$G(\boldsymbol{z}; t) = \int\int \mathrm{D}\boldsymbol{x}\, \mathrm{D}\boldsymbol{z}\, \exp\Big(-\frac{\mathrm{i}}{h}\int_0^t [H(\boldsymbol{x}) - 2(\boldsymbol{x}\,.\,\dot{\boldsymbol{z}}) + 2(\boldsymbol{z}\,.\,\dot{\boldsymbol{z}})]\,\mathrm{d}\tau\Big), \tag{6.6}$$

where the integration is over all continuous trajectories $\boldsymbol{x}(\tau)$ and $\boldsymbol{z}(\tau)$ in the phase space with the boundary condition $\boldsymbol{z}(t) = \boldsymbol{z}$. This is in contrast to the Feynman path integral with the action (1.2), where one should restrict the class of allowed trajectories, correlating the structure of $\boldsymbol{q}(\tau)$ and $\boldsymbol{p}(\tau)$. In the exponent of (6.6) we recognise the phase action functional (2.11). This form of the path integral suggests a method to quantise the generalised mechanics described in the appendix. Probably, one should consider the path integral involving the functional (A.5). A more detailed analysis of the problems mentioned in this section will be presented elsewhere.

In conclusion, note that the phase action formalism may be immediately applied to the dynamics with anticommuting (Grassmann) variables, considered by Berezin and Marinov (1977, and references therein). Roughly speaking, it is sufficient to replace the skew-symmetrical matrix ω by unity, and to write the usual scalar product $\mathrm{i}(\boldsymbol{x}\,.\,\boldsymbol{y})$ instead of $(\boldsymbol{x} \circ \boldsymbol{y})$, defined in (1.5). Meanwhile, the conventional action formalism, apparently distinguishing the coordinate and the momentum, is not applicable to the Grassmann case.

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Appendix. The generalised Hamilton–Jacobi equation and canonical dynamics on a manifold

Consider a manifold with local coordinates z_k, $k = 1, \ldots, n$, and let $\Phi(z; t)$ be the phase action describing the dynamics on the manifold. We start with the Cauchy problem for a partial differential equation, generalising (2.3),

$$\partial\Phi/\partial t = M(\partial\Phi/\partial z - f(z)), \qquad \Phi(z; 0) = 0, \tag{A.1}$$

where $f(z) = \{f^k(z)\}$ is a contragredient vector field on the manifold and $M(y)$ is a regular function on the conjugate manifold. The way to find $\Phi(z; t)$ is as follows (Smirnov 1957 § 114). Solve the set of the characteristic equations

$$\dot{z}_1 = -\partial M(g - f(z))/\partial g^l, \qquad \dot{g}^l = \dot{z}_k\, \partial f^k/\partial z_l, \qquad \dot{\phi} = g^l \dot{z}_l + M(g - f(z)), \tag{A.2}$$

with the initial conditions

$$z(0) = \zeta, \qquad g(0) = 0, \qquad \phi(0) = 0. \tag{A.3}$$

After the natural substitution $y = g - f(z)$, the system is

$$\begin{aligned} &\dot{z}_l = -\partial M(y)/\partial y^l, \qquad \dot{y}^l = \alpha^{lm}(z)\dot{z}_m, \qquad y(0) = -f(\zeta), \\ &\dot{\phi} = y \cdot \dot{z} + f(z)\dot{z} + M(y), \end{aligned} \tag{A.4}$$

where $\alpha^{lm} = \partial f^m/\partial z_l - \partial f^l/\partial z_m$. Suppose that the solution is $z = Z(\zeta, t)$, $y = Y(\zeta, t)$, then

$$\phi = \Psi(\zeta; t) = \int_0^t [(Y + f(Z))\, \partial Z/\partial t + M(Y)]\, d\tau. \tag{A.5}$$

To find the desired result, one has to express the initial point of the trajectory ζ through its end point z; $\Phi(z; t) = \Psi(\zeta(z, t); t)$.

Assume for a moment that the skew-symmetrical field α^{lm} is constant, so that f is linear in z,

$$f^l(z) = -\tfrac{1}{2}\alpha^{lm} z_m, \tag{A.6}$$

(evidently, addition of a gradient to $f(z)$ is of no importance). In this case one is able to exclude y from (A.4), $y = \alpha(z - \zeta/2)$. If α is a non-degenerate matrix, one gets the standard Hamilton equations (1.7) with $\omega = -4\alpha^{-1}$,

$$x = 2z - \zeta = 2\alpha^{-1} y, \qquad x(0) \equiv \xi = \zeta, \qquad H(x) = M(\alpha x/2). \tag{A.7}$$

In this particular case the integral (A.5) is rewritten as (2.11).

In general, equations (A.4) describe the canonical dynamics on the manifold with non-flat symplectic structures. The general construction may be depicted in metaphysical terms. Two manifolds are involved. On one (the Heavens) the Law is established, written by means of the master function $M(y)$, which is an analogue of the Hamiltonian. On another (the Earth) the motion, represented by the function $\Phi(z; t)$, is observed. The correspondence is transferred by the mapping $z \to f(z)$.

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